Physics 212b, Ken Intriligator lecture 5, Jan 24, 2018

• Last time, the Coulomb potential has an additional degeneracy, corresponding to the conserved Runge-Lenz vector[1] $\vec{M} = (2m)^{-1}(\vec{p}\times\vec{L} - \vec{L}\times\vec{p}) - (Ze^2/r)\vec{r}$. This satisfies $[\vec{M}, H] = 0$. Since $\vec{M}$ transforms as a vector, it satisfies $[M_i, L_j] = i\hbar\epsilon_{ijk}M_k$. Also, it satisfies $[M_i, M_j] = -i\hbar\epsilon_{ijk}(2/m)HL_k$. Acting on energy eigenstates, we can replace $H \to E$ and define $\vec{N} = (-m/2E)^{1/2}\vec{M}$, with $[N_i, N_j] = i\hbar\epsilon_{ijk}L_k$. Define $\vec{I} = \frac{1}{2}(\vec{L} + \vec{N})$ and $\vec{K} = \frac{1}{2}(\vec{L} - \vec{N})$, and see that they form decoupled $SU(2)$ representations:

$$[I_i, I_j] = i\hbar\epsilon_{ijk}I_k, \qquad [K_i, K_j] = i\hbar\epsilon_{ijk}K_k, \qquad [I_i, K_j] = 0$$

and $[I_i, H] = [K_i, H] = 0$. So energy eigenstates form representations of $SU(2)_I$ and $SU(2)_K$, labeled by i, k which can be integer or half-integer, and the degeneracy is $(2i+1)(2k+1)$. Also there is another relation: $\vec{I}^2 - \vec{K}^2 = \vec{L}\cdot\vec{N} = 0$, which requires $i = k$. Also

$$\vec{I}^2 + \vec{K}^2 = \tfrac{1}{2}(\vec{L}^2 + \vec{N}^2) = \tfrac{1}{2}(\vec{L}^2 - \frac{m}{2E}\vec{M}^2),$$

with $\vec{M}^2 = \frac{2}{m}H(\vec{L}^2 + \hbar^2) + Z^2e^4$. These imply $2k(k+1)\hbar^2 = \frac{1}{2}(-\hbar^2 - \frac{m}{2E}Z^2e^4)$ and thus $E = -(mZ^2e^4/2\hbar^2)n^{-2}$ with $n = 2k+1$. The degeneracy of $(2k+1)^2$ is thus a degeneracy of n^2. The symmetry completely determines the hydrogen atom energy levels and degeneracy, with no need to solve the Schrodinger equation.

• Since $\vec{L} = \vec{I} + \vec{K}$, the rotation subgroup is a diagonal subgroup of $SU(2)\times SU(2)$; this is standard fact about how 3d rotations are embedded in 4d rotations. Also, $\vec{L}$ is determined by the usual rules for addition of angular momentum from the $\vec{I}$ and $\vec{K}$ representation: ℓ can range from $|i-k|$ to $i+k$. Since $i = k$, this is $0\ldots 2k = n-1$.

• The 3d SHO $V(r) = \frac{1}{2}m\omega^2r^2$ also has an extra symmetry[2] The extra conserved charges are a quadrupole tensor $Q^{\ell=2}_{(ij)}$, i.e. there are 5 extra generators, in addition to the 3 rotation generators, and the fact that they have $\ell = 2$ means that they do not commute with rotations, i.e. the rotation symmetry is a subgroup of a bigger symmetry group[3]

[1] Classically, writing $V = -\kappa/r$, $E = -\kappa/2a$, where a is the semi-major axis (half the distance from the perihelion to the aphelion), and $\vec{L}^2 = \mu\kappa a(1-e^2)$, with $\vec{L}$ perpendicular to the orbit plane, and $|\vec{M}| = \kappa e$ where $e \equiv (a^2 - b^2)^{1/2}/a$ is the eccentricity and $\vec{M}$ points from the origin to the perihelion.

[2] Can again get elliptical classical orbits, with $E = \frac{1}{2}k(a^2 + b^2)$ and $\vec{L}^2 = mKa^2b^2$. Unlike the Coulomb potential, now the origin is at the center of the ellipse, not a focal point.

[3] In quantum field theory, the rotation symmetry $SU(2)$ is a subgroup of the Lorentz group $SO(1,3)$, which is a subgroup of the Poincare group that includes translations. There are strong constraints on embedding this group into a bigger group. Bigger groups include supersymmetry, and conformal symmetry; these are two things that I study in my own research.

The quantum energies are $E_{\vec{n}} = (n_1 + n_2 + n_3 + \frac{3}{2})\hbar\omega$, so taking $n = n_1 + n_2 + n_3$, the degeneracy is $\frac{1}{2}(n+1)(n+2)$. In spherical coordinates, find $\ell = n, n-2, \ldots$, i.e. $n = k + 2\ell$, where $k = 0, 1, 2, \ldots$, e.g. for $n = 3$ get $\ell = 1, 3$ and the degeneracy is $10 = 3 + 7$. It turns out that $SU(2)_{\vec{L}}$ gets enhanced to $SU(3)$, and the n in the energy eigenstate labels the $SU(3)$ representation obtained in a fully-symmetrized tensor product of n fundamentals. (Aside: the strong interactions is based on an exact $SU(3)_C$ color gauge symmetry that rotates quark colors (r,g,b). There is also an approximate $SU(3)_F$ global symmetry that rotates the three lightest quarks (u,d,s).) In terms of the creation and annihilation operators $a_i^\dagger a_j$ have commutation relations that are those of $U(3)$. The additional $U(1)$ generator, which decouples from $SU(3)$, is particle number, which is essentially H itself.

The are two ways to embed the rotation group in $SU(3)$. One is such that, $SU(3) \to SU(2) \times U(1)$, with $\mathbf{3} \to \mathbf{2}_1 + \mathbf{1}_{-2}$; that is not the correct embedding for this case. The correct embedding us such that $SU(3) \to SO(3)$, with $\mathbf{3} \to \mathbf{3}$. This is clear because the $a_i^\dagger$ are in the $\mathbf{3}$ of $SU(3)$, and in the $\mathbf{3}$ vector of the rotation group $SO(3)$, e.g. $a_i^\dagger|0\rangle \sim x_i|0\rangle$ is clearly in the vector of the rotation group. The representation obtained by the symmetrized product in $\mathbf{3} \times \ldots \times \mathbf{3}$ then decomposes under the rotation group according to e.g. $(\mathbf{3} \times \mathbf{3})_S = \mathbf{5} + \mathbf{1}$ etc, agreeing with $n = k + 2\ell$.