140a Lecture 17, 3/7/19

⋆ Week 9 reading: Blundell+Blundell, chapters 29, 23, 24

• Continue where we left off last time: consider e.g. a 1-state system, where the one state has energy E. Consider the grand partition function $\mathcal{Z} = \sum_n e^{n\beta(\mu-E)}$, where n is the occupation number. Then $\langle n\rangle = -k_B T\partial \ln \mathcal{Z}/\partial E$.

For Fermions, the only allowed values in $\sum_n$ are $n = 0, 1$, so $\mathcal{Z}_{fermions} = 1 + e^{\beta(\mu-E)}$. For bosons we sum the geometric series over all $n = 0, \ldots \infty$ to get $\mathcal{Z}_{bosons} = (1 - e^{-\beta(\mu-E)})^{-1}$. Both can be written as giving $\ln \mathcal{Z}_{F,B} = \pm \ln(1 \pm e^{\beta(\mu-E)})$. Get $\langle n\rangle = (e^{\beta(E-\mu)} \pm 1)^{-1}$. The distribution function is the mean occupation of a single particle state with energy E and is given by $f_{FD}(E) = (e^{\beta(E-\mu)} + 1)^{-1}$ and $f_{BE}(E) = (e^{\beta(E-\mu)} - 1)^{-1}$.

For $\beta(E - \mu) \gg 1$, the occupation number is dilute and the statistics become unimportant: both distributions approach the Boltzmann distribution $e^{-\beta(E-\mu)}$. This is called the classical limit.

For Fermions in the limit where $\beta\mu \gg 1$, note that for $f_{FD} \approx \Theta(\mu - E)$ where $\Theta(x)$ is a step function, i.e. Fermions occupying up to $E \leq \mu$, filling up to the Fermi surface. For bosons get $f_{BE} \to \infty$ as $E \to \mu$ (Bose condensation).

For a system with energy levels E_i and occupation numbers n_i, and degeneracy g_i the grand partition function is $\mathcal{Z} = \prod_i \sum_{\{n_i\}} e^{n_i\beta(\mu-E_i)}$. For Fermions (+) and Bosons (-) get $\ln \mathcal{Z} = \pm \sum_i \ln(1 \pm e^{\beta(\mu-E_i)})$ and then $\langle n_i\rangle = -\frac{1}{\beta}\partial \ln \mathcal{Z}/\partial E_i = (e^{\beta(E_i-\mu)} \pm 1)^{-1}$.

We previously discussed the microcanonical description $S(U, N, ...) = k \ln \Omega(U, N, ...) \approx k \ln \omega_{max}$ where $\Omega(U, N) = {\sum_{\{n_i\}}}' \omega(\{n_i\})$, with $\omega(\{n_i\})_{B.E.} = \prod_i \frac{(n_i+g_i-1)!}{n_i!(g_i-1)!}$ and $\omega(\{n_i\})_{F.D.} = \prod_i \frac{g_i!}{n_i!(g_i-n_i)!}$. In the HW exercise you used these to compute S and maximize in the n_i to connect to the above expressions from the microcanonical description (enforce the constraints $N = \sum_i n_i$ and $U = \sum_i n_i E_i$ being held fixed via Lagrange multipliers α and β, then solve for them in terms of T and μ).

• Next topic: a thermal collection of photons in a box = blackbody spectrum. Classically, we consider electromagnetic waves and compute the energy momentum tensor to find the energy and radiation pressure on the walls. The solutions of the wave equation in the box, subject to say conducting boundary conditions at the wall, lead to Fourier mode numbers $2V d^3\vec{k}/(2\pi)^3$ (where the 2 is for polarizations) and $\omega = ck$. Integrating over solid angle, get $g(\omega)d\omega = V(4\pi)\omega^2 d\omega/(2\pi c)^3$. The classical equipartition theorem suggests that each frequency ω has energy $k_B T$ (like a SHO, with one kinetic and one vibrational restoring d.o.f.). This leads to a clearly wrong answer for high ω modes. It led

Planck to introduce $\hbar$, to guess the answer before anyone understood QM and what his fix really meant.

Now consider it in QM, where the light is replaced with photons (=quanta of a quantum photon field). Recall that photons have energy $E = \hbar\omega$ and $\vec{p} = \hbar k$, with $\omega = ck$. The energy density of a gas of photons of frequency ω is $u = U/V = n\hbar\omega$ where $n = N/V$ is the photon density. The pressure of a gas of photons is found from the impulse $\Delta p_\perp$ times the flux, which for photons gives $p = u/3$, with the $1/3$ from averaging $\cos\theta$ over the half of the solid angle that is hitting the wall. We got this in lecture 3 for massive particles and it is similar here, with $p = \frac{N}{V}\langle p_z v_z\rangle = \frac{N}{3V}\langle \vec{p}\cdot\vec{v}\rangle$. For the non-relativistic case we have $E = \frac{1}{2}\vec{p}\cdot\vec{v}$, whereas for photons $E = \vec{p}\cdot\vec{v}$, so we get $p = u/3$ for photons. Another way to obtain this is from $p = \sum_i \frac{1}{e^{\beta\epsilon_i}-1}(-\frac{\partial\epsilon_i}{\partial V})$ with $\epsilon_i = \hbar c(2\pi/V^{1/3})\sqrt{n_x^2+n_y^2+n_z^2}$ so $\partial\epsilon_i/\partial V = -\frac{1}{3}\epsilon_i/V$, giving $p = u/3$. Also note that $p = u/3$ fits with the energy momentum tensor being traceless which is the case for scale invariant theories (fitting with photons being massless).

The number of photons hitting a unit area of the container wall per second is $\Phi = \frac{1}{4}nc$ and thus the power incident on the wall per unit area is $P = \hbar\omega\Phi = \frac{1}{4}uc$.

The thermodynamic relations $(\partial U/\partial V)_T = T(\partial S/\partial V)_T - p = T(\partial p/\partial T)_V - p$ becomes $u = \frac{1}{3}(T(\partial u/\partial T)_V - u)$ which leads to $4dT/T = du/u$ and thus $P = \frac{1}{4}uc = \sigma T^4$. Let's now show how to get this, and derive the value of the Stefan-Boltzmann constant σ.