2/3/20 Lecture outline

⋆ **Reading: Sections 10.4 through 10.8**

• Moment of inertia tensor $I_{jk} \equiv \sum_a m_a(\vec{r}_a^2\delta_{jk} - r_j r_k) = I_{kj}$ enters in $L_j = I_{jk}\omega_k$ and $T_{rot} = \frac{1}{2}I_{jk}\omega_j\omega_k$ (sum repeated indices $k = 1, 2, 3$. Under a rotation, vectors transform as $\vec{\omega} \to \vec{\omega}' = R\vec{\omega}$ where $R_{k'j}$ is a 3×3 rotation matrix (which is an orthogonal matrix $R^T = R^{-1}$, which ensures that $\vec{\omega}' \cdot \vec{\omega}' = \vec{\omega} \cdot \vec{\omega}$. A tensor I_{jk} transforms as a vector for each index, which can be written as $I \to I' = RIR^T$. Such a rotation can be used to diagonalize I_{jk}, and the rotation is given by the matrix of eigenvectors: $R_{aj} = \psi^a_j$ (each row a has columns given by an eigenvector).

• The eigenvectors $\vec{\psi}^a$ of the inertia tensor are called the principal axes, and the eigenvalues λ_a are called the principal moments: $\vec{L} = \lambda\vec{\omega}$. We can find three orthogonal eigenvectors $\vec{\psi}_{i=1,2,3}$ and write I in this basis as a diagonal matrix with the three eigenvalues $\lambda_{i=1,2,3}$ along the diagonal. Recall that $\mathrm{Tr}I$ and $\det I$ are invariant under $I \to RIR^{-1}$, so they can be written in terms of the eigenvalues as $\mathrm{Tr}I = \sum_{i=1}^3 \lambda_i$ and $\det I = \prod_{i=1}^3 \lambda_i$.

Let's agree to call the eigenvalues $\lambda_j = I_j$.

• Recall the example of a cube of mass M and side length a. If it's rotating around its center, easily get $\mathbf{I} = \frac{1}{6}\mathbf{1}$: it is diagonal, with equal eigenvalues. Now instead consider rotation around a corner, taking the corner at the origin. Get I_{corner} either from doing the integrals or the parallel axis theorem: $I_{j=k}\frac{2Ma^2}{3}$ and $I_{j\neq k} = -\frac{Ma^2}{4}$. By symmetry, clearly one of the principal axes is along the diagonal, so $\vec{\psi}_1 = \frac{1}{\sqrt{3}}(1, 1, 1)$, which has principal moment eigenvalue $I_1 = Ma^2/6$. The other two principal axes are perpendicular and here, because of the symmetry, they have the same eigenvalue, $I_2 = I_3$. The original I must have trace equal to $I_1 + I_2 + I_3$ and determinant equal to $I_1I_2I_3$. Indeed, find $I_2 = I_3 = \frac{11}{12}Ma^2$.

• Planar mass distribution (in the plane $z = 0$) has $I_{xz} = I_{yz} = 0$ and $I_{xx} + I_{yy} = I_{zz}$. One of the eigenvectors is $I_3 = I_{zz}$. The other two eigenvectors are those of the 2×2 block and they satisfy $I_1 + I_2 = I_3$ and $I_2I_3 = I_{xx}I_{yy} - I_{xy}^2$.

• Consider a top that is spun along a principal axis, say $\vec{\omega} = \vec{\psi}_3$, so $\vec{L} = I_3\vec{\omega}$. The external torque from gravity acts on the CM as $\vec{\Gamma}^{ext} = \vec{R} \times M\vec{g}$ and then $\frac{d}{dt}\vec{L} = \vec{R} \times M\vec{g}$. Assuming the top is symmetric we have $\vec{R} = \frac{R}{\omega}\vec{\omega}$ and thus $\dot{\vec{L}} = \frac{MgR}{I\omega}\hat{z} \times \vec{L} \equiv \vec{\Omega} \times \vec{L}$ with $\vec{\Omega} = \frac{MgR}{I\omega}\hat{z}$ the angular velocity of precession. (Note units are indeed s^{-1}.)