2/5/20 Lecture outline

⋆ **Reading: Sections 10.4 through 10.8**

• Last time: top that is spun along a principal axis, say $\vec{\omega} = \vec{\psi}_3$, so $\vec{L} = I_3\vec{\omega}$. The external torque from gravity acts on the CM as $\vec{\Gamma}^{ext} = \vec{R} \times M\vec{g}$ and then $\frac{d}{dt}\vec{L} = \vec{R} \times M\vec{g}$. Assuming the top is symmetric we have $\vec{R} = \frac{R}{\omega}\vec{\omega}$ and thus $\dot{\vec{L}} = \frac{MgR}{I\omega}\hat{z} \times \vec{L} \equiv \vec{\Omega} \times \vec{L}$ with $\vec{\Omega} = \frac{MgR}{I\omega}\hat{z}$ the angular velocity of precession. (Note units are indeed s^{-1}.)

• It is useful to go to the non-inertial, rotating frame whose basis vectors are the principal axes. Recall that for any vector $\dot{\vec{Q}}|_{space} = \dot{\vec{Q}}|_{body} + \vec{\omega} \times \vec{Q}$ where "space" refers to an inertial frame that is fixed in the lab, and "body" refers to a non-inertial frame that is fixed on the rotating body, so $\vec{\Omega} \to \vec{\omega}$.

Apply this to the case of angular momentum to get Euler's equation:

$$\frac{d\vec{L}}{dt}|_{space} = \vec{\Gamma}_{ext} = \frac{d\vec{L}}{dt}|_{body} + \vec{\omega} \times \vec{L}.$$

Use this and $L_j = I_{jk}\omega_k$ to determine the dynamical rotation $\vec{\omega}(t)$ of the body.

In the body frame, we can take $\vec{R} = 0$ so e.g. for $\vec{\Gamma}^{ext} = \vec{R} \times \vec{F}_{ext}$, get $\vec{\Gamma}^{ext} = 0$.

• Use the principal axis basis, so $\vec{\omega} = \sum_{i=1}^{3} \omega_i\vec{\psi}_i$ and $\vec{L} = \sum_i I_i\omega_i\vec{\psi}_i$, Euler's equations are

$$I_i\dot{\omega}_i = \sum_{jk} \epsilon_{ijk} I_j\omega_j\omega_k + \vec{\Gamma}_k^{ext}.$$

The equations are generally complicated to solve, even for case $\vec{\Gamma}^{ext} = 0$. A special case: if $\omega_1 = \omega_2 = 0$, then get $\omega_3 = \omega_0$ is a constant.

Now study small variations from this case, $\vec{\omega} = \omega_0\vec{\psi}_3 + \delta\vec{\omega}$ and linearize in $\delta\omega_i$. Get $\dot{\delta\omega_3} \approx 0$

$$I_1\delta\dot{\omega}_1 \approx (I_2 - I_3)\omega_0\delta\omega_2, \qquad I_2\delta\dot{\omega}_2 \approx (I_3 - I_1)\omega_0\delta\omega_1.$$

Combine to get:

$$\frac{d^2}{dt^2}\omega_{i=1,2} = -\Omega^2\omega_{i=1,2}, \qquad \Omega^2 = \frac{(I_3 - I_2)(I_3 - I_1)}{I_1 I_2}\omega_0^2.$$

The equations are stable oscillations if $\Omega^2 > 0$ and are exponentials if $\Omega^2 < 0$. Tennis racquet theorem: rotation around the principal axis with largest or smallest moment is stable, whereas rotation around the axis with middle moment of inertia is unstable.

• Torque free with $I_1 = I_2$: free precession. Euler's equations give $\dot{\omega}_3 = 0$, so ω_3 is a constant, and $\dot{\omega}_1 = -\Omega_p\omega_2$ and $\dot{\omega}_2 = \Omega_p\omega_1$ with $\Omega_p = (I_3 - I_1)\omega_3/I_1$.