2/7/20 Lecture outline

⋆ **Reading: Finish Chapter 10**

• Last time: Euler's equations

$$I_i\dot{\omega}_i = \sum_{jk}\epsilon_{ijk}I_j\omega_j\omega_k + \vec{\Gamma}^{ext}_k$$

for the case with $\vec{\Gamma}^{ext} = 0$ (e.g. with $\Gamma^{ext} = \vec{R}\times\vec{F}_{ext}$ upon taking $\vec{R} = 0$ in the body frame). We saw that if $\omega_1 = \omega_2 = 0$, then $\omega_3 = const.$ is a solution, and that small deviations $\delta\omega_{1,2}$ oscillate with angular frequency Ω given by $\Omega^2 = (I_3 - I_2)(I_3 - I_1)\omega_3^2/I_1I_2$, so it is stable ($\Omega^2 > 0$) if I_3 is the largest or smallest eigenvalue, but not if it is the middle eigenvalue. We next considered the special case $I_1 = I_2$ (axial symmetric object) beyond the small $\omega_{1,2}$ limit (note that $\Omega^2 > 0$ if $I_1 = I_2$). Euler's equations for $I_1 = I_2$ give $\dot{\omega}_3 = 0$, so ω_3 is a constant, and $\dot{\omega}_1 = -\Omega_p\omega_2$ and $\dot{\omega}_2 = \Omega_p\omega_1$ with $\Omega_p = (I_1 - I_3)\omega_3/I_1$. Use $\eta = \omega_1 + i\omega_2$ to write these as $\dot{\eta} = -i\Omega_b\eta$ so $\eta = \omega_0 e^{-i\Omega_p t}$. Thus $\vec{\omega} = (\omega_0\cos\Omega_p t, -\omega_0\sin\Omega_p t, \omega_3)$ and $\vec{L} = (I_1\omega_1, I_1\omega_2, I_3\omega_3)$. See that $\vec{\omega}(t)$ and $\vec{L}(t)$ and $\vec{\psi}_3$ all line in a plane with constant angle between them, and $\vec{L}$ and $\vec{\omega}$ precess around $\vec{\psi}_3$ at rate Ω_p. For the earth $I_1 = I_2 \approx (299/300)I_3$. So $\Omega_p \approx \omega_3/300$ so the rotation would precess in about 300 days in some approximation – this is the Chandler wobble and the precession is actually around 400 days (presumably because of the oceans, so the earth is not perfectly rigid).

• Euler angles: a general rotation is parameterized by three angles (corresponding to the fact that there are three $\vec{L}$ generators of rotation). Can get a general rotation as follows: $R = R_z(\psi)R_y(\theta)R_z(\phi)$, where $R_z(\phi) = \begin{pmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}$, etc. In words, the steps are (1) Rotate around the $\hat{z}$ axis by angle ϕ. (2) Rotate around the new $\vec{e}_2'$ axis by angle θ. Now the body axis $\vec{e}_3''$ is a vector with polar angles θ and ϕ, so (3) rotate around $\vec{e}_3$ by an angle ψ. This defines the Euler angles.

Use this process to go from the initial inertial basis $\hat{x}, \hat{y}, \hat{z}$ to the eigenbasis vectors $\vec{\psi}^1$, $\vec{\psi}^2$, $\vec{\psi}^3$ on the body. Then $\vec{\omega} = \dot{\phi}\hat{z} + \dot{\theta}\hat{e}_2' + \dot{\psi}\vec{\psi}^3$, where $\hat{z} = \cos\theta\vec{\psi}_3 - \sin\theta\hat{e}_1'$. Get $\omega_3 = \vec{\omega}\cdot\vec{\psi}^3 = \dot{\psi} + \dot{\phi}\cos\theta$ and $L_3 = I_3\omega_3$. More generally, it is convenient to use a basis of $\hat{e}_1'$ and $\hat{e}_2'$ which are the intermediate (x, y) axes, along with $\vec{\psi}_3$, e.g. $\vec{\omega} = (-\dot{\phi}\sin\theta)\hat{e}_1' + \dot{\theta}\hat{e}_2' + (\dot{\psi} + \dot{\phi}\cos\theta)\vec{\psi}_3$ and $\vec{L} = (I_1\omega_1, I_2\omega_2, I_3\omega_3) = (-I_1\dot{\phi}\sin\theta)\hat{e}_1' + I_2\dot{\theta}\hat{e}_2' + I_3(\dot{\psi} + \dot{\phi}\cos\theta)\vec{\psi}_3$. The kinetic energy is $T = \frac{1}{2}I_1(\dot{\phi}\sin\theta)^2 + \frac{1}{2}I_2\dot{\theta}^2 + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2$.

• External torque free symmetric top (e.g. a dreidel), $I_1 = I_2$, with one point fixed. The axis of rotation is ψ_3 and then

$$L = \frac{1}{2}I_1(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 - MgR\cos\theta.$$

Get $p_\phi = \frac{\partial L}{\partial \dot{\phi}} = (I_1 \sin^2\theta + I_3\cos^2\theta)\dot{\phi} + I_3\cos\theta\dot{\psi} = L_z$ and $p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3(\cos\theta\dot{\phi} + \dot{\psi}) = L_3$ are constants of the motion, as is

$$E = H = \frac{1}{2}I_1\dot{\theta}^2 + U_{eff}(\theta), \qquad U_{eff}(\theta) = \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + \frac{p_\psi^2}{2I_3} + MgR\cos\theta.$$

Find that $U_{eff}(\theta)$ has a minimum and two turning points where $E = U_{eff}$, so θ oscillates – this is called nutation. One can solve for $\dot{\phi}$ in terms of the conserved angular momenta: $\dot{\phi} = (p_\phi - p_\psi\cos\theta)/I_1\sin^2\theta$. If $|p_\phi| > |p_\psi|$, then $\dot{\phi} \neq 0$ so the precession is always in the same direction. In the other case, it's possible for the precession direction to change.

Suppose that θ is sitting at the minimum of $U_{eff}(\theta)$, then θ is a constant and then it follows from the above equation for $\dot{\phi}$ that $\dot{\phi}$ is a constant, $\dot{\phi} \equiv \Omega$. This is steady precession. The θ equations of motion show that $\dot{\theta} = 0$ requires minimizing $U_{eff}(\theta)$ and this implies that Ω must satisfy the quadratic equation $I_1\Omega^2\cos\theta - I_3\omega_3\Omega + MgR = 0$. For a rapidly spinning top, the two roots are $\Omega_{small} \approx MgR/I_3\omega_3$ and $\Omega_{large} \approx I_3\omega_3/I_1\cos\theta$ (this latter one is the free precession expected in the absence of torques).