2/19/20 Lecture outline

⋆ **Reading: Taylor sections 15.4 to 15.14.**

• Last week we discussed how $x^\mu = (ct, \vec{x})$ behaves under Lorentz transformations, e.g. boosts along the x axis by some relative velocity v_{rel}. Let's write this as $x^{\mu'} = \Lambda^{\mu'}_\nu x^\nu$. Likewise, $dx^\mu = \Lambda^{\mu'}_\nu dx^\nu$. Aside: this gives $v/c = \frac{dx}{cdt} = \frac{\beta_{rel}\gamma_{rel}dt' + \gamma_{rel}dx'}{\gamma_{rel}cdt' + \beta_{rel}\gamma_{rel}dx'} = \frac{\beta_{rel}+\beta'}{1+\beta_{rel}\beta'}$, where $\beta' = dx'/dt/$, which is the relativistic velocity addition formula.

• x^μ and dx^μ are examples of 4-vectors. All 4-vectors $a^\mu = (a^0, \vec{a})$ transforms the same way under Lorentz transformations, $a^{\mu'} = \Lambda^{\mu'}_\nu a^\nu$, where $\Lambda^{\mu'}_\nu$ can be a rotation (which transforms $\vec{a}$ by the usual matrix with $\cos\theta$ and $\sin\theta$, preserving dot products) or a boost along any of the three space directions, which mixes a^0 and the component of $\vec{a}$ along the boost direction by the usual matrix with γ_{rel} and $\beta_{rel}\gamma_{rel}$. The invariant interval can be written as $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$ where $\eta_{\mu\nu} \equiv diag(1, -1, -1, -1)$ (this is called the mostly minus convention, and the book instead uses the mostly plus convention). $\eta_{\mu\nu}$ is a two index (symmetric) tensor and any tensor $X_{\mu\nu}$ transforms like a 4-vector for each index, so $X_{\mu'\nu'} = \Lambda_{\mu'}{}^{\mu}\Lambda_{\nu'}{}^{\nu}$, which we can write in matrix form as $X' = \Lambda X \Lambda^T$. A special property of η is that it is preserved by Loretnz transformations: $\eta' = \Lambda\eta\Lambda^T = \eta$. This ensures that $ds^{2\prime} = ds^2$. Indeed, for any 4-vectors a^μ and b^μ, contracting the indices with $\eta_{\mu\nu}$ gives a Lorentz invariant: $a \cdot b \equiv a^0b^0 - \vec{a}\cdot\vec{b} \equiv a^\mu b_\mu$ where $b_\mu = \eta_{\mu'\nu}b^\nu = (b_0, -\vec{b})$. All Lorentz related observers see the same value: $a \cdot b = a' \cdot b'$; the invariant interval $ds^2 = dx^\mu dx_\mu$ is a special case of this. The proper time $d\tau = \sqrt{ds^2/c^2} = dt/\gamma$ is Lorentz invariant.

• In physics we meet many vectors, and the following ones extend into 4-vectors: position, velocity, momentum, force, current density, wavenumber, gradient. (Some vectors instead combine into 4-tensors: $\vec{E}$ and $\vec{B}$, and the energy flux, as we will discuss later.)

• Since dx^μ is a 4-vector, and $d\tau = dt/\gamma$ is a Lorentz scalar, $u^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\frac{dt}{d\tau} = \gamma(c, \vec{v})$ is a 4-vector version of velocity. Note that $u_\mu u^\mu = c^2$.

• Addition of velocities, again: suppose that someone in the $'$ frame throws an object with velocity $\vec{v}' = v_1\hat{x}$. What is the velocity of the object as seen in the original frame. One way to analyze this is to go to the $''$ frame, where the rock is at rest, and ask what is the Lorentz transformation back to the lab frame. Thus $\begin{pmatrix} \gamma_T & \beta_T\gamma_T \\ \beta_T\gamma_T & \gamma_T \end{pmatrix} = \begin{pmatrix} \gamma_1 & \beta_1\gamma_1 \\ \beta_1\gamma_1 & \beta_1 \end{pmatrix}\begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \beta \end{pmatrix}$, which gives the velocity addition formula $\beta_T = (\beta_1 + \beta)/(1 + \beta_1\beta)$. This is equivalent to the comments in a previous lecture about rapidities adding, where $\phi \equiv \tanh^{-1}(\beta)$ and then $\phi_T = \phi + \phi_1$.

Instead, we can get this result by using the velocity 4-vector: use the fact that the observer in the $'$ frame sees 4-velocity $u^{\mu'} = (\gamma_1, \gamma_1\beta_1)$. Then we transform that to the lab frame by the usual Lorentz transformation matrix to get $u^0 = \frac{dt}{d\tau} = \gamma\gamma_1(1+\beta\beta_1)$ and $u^1 = \frac{dx}{d\tau} = \gamma\gamma_1(\beta+\beta_1)$. Then the velocity as seen in the lab frame is $dx/dt = u^1/u^0 = (\beta+\beta_1)/(1+\beta\beta_1)$, as above. Note that $u^{2'} = u^2$ for boosts along the x axis but that $v'_y = \frac{dy}{\gamma(dt - v_{rel}dx/c^2)} = v_y\gamma^{-1}(1 - v_x V_{rel}/c^2)^{-1}$.

• Energy and momentum combine into a 4-vector $p^\mu = (E/c, \vec{p})$, with $p_\mu p^\mu = (mc)^2$.