3/4/20 Lecture outline

⋆ **Reading: Taylor chapter 16.1, 16.2, 16.3**

• We are starting a new topic: continuum mechanics, aka field theory. In the mechanics of point particles, we want to solve for particle positions $q_a(t)$. The action is $S = \int dt L(q_a, \dot{q}_a)$ and $\delta S = 0$ leads to the equations of motion for $q_a(t)$. In field theory, the dynamical quantities depend on both t and some space coordinates. For example, the displacement of a string from equilibrium is a field $\psi(t, x)$. Then $S = \int dt dx L(\psi, \partial_t \psi, \partial_x \psi)$.

• E&M is an example of a field theory. Can get Maxwell's equations from least action in the field A_μ with $S_{fields} = \int d^t d^3 x \mathcal{L}$ with $\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} A_\mu J^\mu = \frac{1}{8\pi}(\vec{E}^2 - \vec{B}^2) - \rho\phi + \frac{1}{c}\vec{A} \cdot \vec{J}$. Note that it is relativistically invariant. Recall that $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. Varying the action $\delta S = \int d^4 x \delta A_\mu (\frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu \frac{\partial \mathcal{L}}{\partial_\nu A_\mu})$ gives field Euler Lagrange equations. For $\mathcal{L}_{E\&M}$, this gives Maxwell's equations: $\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\mu$.

The Lorentz force law, $\frac{dp^\mu}{d\tau} = f^\mu = \frac{q}{c} F^{\mu\nu} u_\nu$, comes from $S = \ldots - \frac{q}{c} \int A_\mu dx^\mu$. For point charges q_a, $J^\mu = \sum_a q_a \frac{dx^\mu}{dt} \delta^3(\vec{x} - \vec{x}_a(t))$. The $A_\mu J^\mu$ term in the Lagrangian density then integrates to give the $q_a A_\mu dx^\mu$ term in the action for the charged particles.

• In empty space, $J^\mu = 0$, the solutions of Maxwell's eqns satisfy a wave equation $\partial_\mu \partial^\mu \psi = 0$, where $\partial_\mu \partial^\mu = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2$ is Lorentz invariant and ψ is $\vec{E}$ or $\vec{B}$.

Example of solutions of the wave equation $\psi = f(x - vt)$ for $v = c$ and any function (plane wave moving along the x axis at $v = c$). $\psi = Ae^{-ik_\mu x^\mu}$ if $k^2 = 0$, i.e. $\omega = ck$.

• Fields carry energy and momentum. In E&M, energy and momentum can go between the fields and the particles, and only the totals are conserved. The field energy density is $\mathcal{H}_{field} = \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2)$: the field energy in a region is $E_{field} = \int_V d^3 x \mathcal{H}$. The field momentum in a region is $c\vec{P}_{field} = \frac{1}{4\pi} \int_V d^3 x (\vec{E} \times \vec{B})$. In relativistic notation, we have $P^\mu_{field} = \int d^3 x T^{\mu 0}_{field}$ where $T^{\mu\nu} = T^{\nu\mu}$ is the stress-energy-momentum tensor. It transforms like a symmetric, two-index tensor under Lorentz transformations $T^{\mu'\nu'} = \Lambda^{\mu'}{}_\rho \Lambda^{\nu'}{}_\sigma T^{\rho\sigma}$. Just as electric charge $Q = \int_V d^3 x J^0/c$ is conserved if $\partial_\mu J^\mu = 0$, likewise $P^\mu = \int d^3 x T^{\mu 0}$ is conserved if $\partial_\nu T^{\mu\nu} = 0$. For the case of E and M, $T^{\mu\nu}_{field} = \frac{1}{4\pi} F^{\mu\lambda} F^\nu{}_\lambda + \frac{1}{16\pi} \eta^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}$. The conserved quantity is $T^{\mu\nu}_{total} = T^{\mu\nu}_{matter} + T^{\mu\nu}_{field}$, where $T^{\mu\nu}_{matter} = \sum_a \frac{p^\mu_a p^\nu_a}{E_a/c^2} \delta^3(\vec{x} - \vec{x}_a(t))$ and $\partial_\mu T^{\mu\nu}_{matter} = -\partial_\mu T^{\mu\nu}_{field} = \frac{1}{c} F^{\nu\kappa} J_\kappa$. Integrating fits with the $\mu = 0$ component of $\frac{dp^\mu}{d\tau} = f^\mu = \frac{q}{c} F^{\mu\nu} u_\nu$ power loss for particles $\frac{dp^0}{d\tau} = \frac{q}{c} \vec{u} \cdot \vec{E}$ using $F^{i0} = E^i$.

• Just for fun: general relativity is another example of a field theory. Replace $\eta_{\mu\nu}$ with a dynamical spacetime metric $g_{\mu\nu}(x)$. Einstein's equations state that $T^{\mu\nu}$ acts as a source for derivatives of the metric (don't worry about the details): $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$. Get it from least action for field $g_{\mu\nu}$ via $S_{EH} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R$. The details are beyond the scope of this class – here just illustrating some generalizations of the things that we're discussing. E.g. the solution for a mass M at the origin is the Schwarzschild metric:

$$ds^2 = (1 - (2GM/r))(cdt)^2 - (1 - (2GM/r))^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Note that $d\tau_{A,B} = \sqrt{1 - (2GM/r_{A,B}}dt$ so the person A with $r_A > r_B$ up ages more than person B; atomic clocks are sufficiently precise to measure this difference, even for ~ 1 m high differences in the earth's gravity field. If person A drops photons with $\omega_{emit} = 2\pi/\tau_A$, person B receives them with $\omega_B = 2\pi/\tau_B$, and finds that they are gravitationally blueshifted from their fall. Pound and Rebka measured this (using the Mossbauer effect) in 1949 by dropping photons 20 meters in the Harvard physics building.

Black holes if $r_{object} < 2GM$. The sign change of the terms at the horizon, and the interpretation.