1/13/20 Lecture outline

⋆ **Reading: Taylor Chapter 8.**

• Last time: orbit equations have solution $r = r(t)$ determined from $\mu\frac{d^2r}{dt^2} = -\frac{d}{dr}U_{eff}$ with $U_{eff} \equiv \frac{\ell^2}{2\mu r^2} + U$, or $\frac{1}{2}\mu\dot{r}^2 + U_{eff} = E$, and $\phi = \phi(t)$ determined from $\ell = \mu r^2\dot{\phi}$.

Let's study the shape of the trajectory rather than the t dependence. Eliminating the parameter t, we can solve for $r = r(\phi)$. To do this, use

$$\frac{d}{dt} = \dot{\phi}\frac{d}{d\phi} = \frac{\ell}{\mu r^2}\frac{d}{d\phi} = \frac{\ell u^2}{\mu}\frac{d}{d\phi},$$

where $u = 1/r$ is introduced for convenience. So

$$\frac{dr}{dt} = -\frac{\ell}{\mu}\frac{du}{d\phi}, \qquad \frac{d^2r}{dt^2} = -\frac{\ell^2u^2}{\mu^2}\frac{d^2u}{d\phi^2},$$

and the r EOM becomes (with $F(r) = -dU/dr$)

$$u''(\phi) + u + \frac{\mu}{\ell^2u^2}F(r) = 0.$$

Energy conservation is

$$E = \tfrac{1}{2}\mu\left(\frac{\ell}{\mu r^2}\frac{dr}{d\phi}\right)^2 + \frac{\ell^2}{2\mu r^2} + U(r),$$

which we can use to solve for $dr/d\phi$, and then integrate the equation to obtain

$$\phi - \phi_0 = \int_{r_0}^{r}\frac{\ell dr/r^2}{\sqrt{2\mu(E - U_{eff}(r))}}$$

Example: for a free particle, $U(r) = F(r) = 0$ and the solution of the EOM is $u(\phi) = r_0^{-1}\cos(\phi - \delta)$, the equation of a straight line, good, with $E = \frac{\ell^2}{2\mu}r_0^{-2} = \frac{1}{2}\mu v_0^2$.

• For general $U(r)$, circular orbit at points $r = r_0$ where $U'_{eff}(r_0) = 0$. Stable if $U''(r_0) > 0$. Then consider nearly circular orbits by expanding $r = r_0 + \epsilon(t)$ and find

$$\frac{d^2\epsilon}{dt^2} = -\frac{U''_{eff}(r_0)}{\mu}\epsilon \equiv -\omega^2\epsilon,$$

which has solution $\epsilon = \epsilon_0\cos\omega t$, with $\omega \equiv \sqrt{U''_{eff}(r_0)/\mu}$ the frequency of oscillation about $r = r_0$. For circular orbits, $u = u_0$ =constant. For nearly circular orbits, we can write $u = u_0 + \delta(\phi)$ and expand the above to find an equation for $\delta(\phi)$. Let's instead write it in terms of the original variable r, so $r = r_0 + \eta(\phi)$ and then plug into the equation above to find

$$\frac{d^2\eta}{d\phi^2} = -\beta^2\eta,$$

where

$$\beta^2 \equiv 3 - \frac{\mu r_0^4}{\ell^2} F'(r_0).$$

A solution is $\eta(\phi) = \eta_0 \cos\beta\phi$. The maximum is chosen at $\phi_n = 2\pi n/\beta$.

• Kepler orbits: $U(r) = -k/r$, so $F(r) = -k/r^2$. (Sign is chosen so that $k > 0$ corresponds to an attractive force). Get

$$u''(\phi) = -u(\phi) + k\mu/\ell^2,$$

which is like the free particle, if we substitute $w = u - k\mu/\ell^2$, so

$$r(\phi) = \frac{c}{1 + \epsilon\cos\phi}, \qquad c \equiv \frac{\ell^2}{k\mu}. \tag{1}$$

where ϵ is a constant, which can be written in terms of the energy as

$$\epsilon = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}.$$

So $\epsilon < 1$ gives bounded orbits, and $\epsilon > 1$ gives unbounded orbits. For $\epsilon < 1$ the equation is an ellipse (with special case being a circle for $\epsilon = 0$). For $\epsilon > 1$ it is a hyperbola. For $\epsilon = 1$ it is a parabola.

• For $E < 0$ (bound orbits) get $\epsilon < 1$, and the above conic section is an ellipse. The ellipse has major and minor semi-axes given by

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad a = \frac{c}{1-\epsilon^2},\ b = \frac{c}{\sqrt{1-\epsilon^2}},\ d = a\epsilon.$$

The ellipse has foci at $(0,0)$ and $(-2d, 0)$ and is the set of points such that the sum of the distances to each foci is $2a$. Then

$$\frac{b}{a} = \sqrt{1-\epsilon^2}, \qquad r_{min} = \frac{c}{1+\epsilon}, \qquad r_{max} = \frac{c}{1-\epsilon}.$$

Also,

$$r_{min} = \frac{c}{1+\epsilon} = a(1-\epsilon), \quad r_{max} = \frac{c}{1-\epsilon} = a(1+\epsilon).$$

Since $1 - \epsilon^2 = -2E\ell^2/\mu k^2 = 2|E|\ell^2/\mu k^2$ we have

$$a = \frac{k}{2|E|}, \qquad b = \frac{\ell}{\sqrt{2\mu|E|}}.$$

The energy is

$$E = U_{eff}(r_{min}) = -\frac{k}{r_{min}} + \frac{\ell^2}{2\mu r_{min}^2} = \frac{k^2\mu}{2\ell^2}(\epsilon^2 - 1).$$

The period of revolution is given by recalling $dA/dt = \ell/2\mu$ (Kepler's 2nd law), so the period is $\tau = A/\dot{A} = 2\pi ab\mu/\ell$ so

$$\tau = 2\pi a^{3/2}\sqrt{\frac{\mu}{k}} = \pi k\sqrt{\frac{\mu}{2|E|^3}}.$$

Note that the period is uniquely determined by the energy.

For a comet or planet orbiting the sun, $k = Gm_1m_2 \approx G\mu M_s$ so $\tau^2 \approx 4\pi^2a^3/GM_s$; Kepler's 3rd law.

• For $\epsilon = 1$, get $y^2 = c^2 - 2cx$, a parabola. For $\epsilon > 1$, get hyperbola:

$$\frac{(x-\delta)^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1.$$

• Orbit change by tangential thrust at perigee. Initial and final orbits have the same perigee

$$r_{min} = \frac{c_1}{1+\epsilon_1} = \frac{c_2}{1+\epsilon_2}.$$

The velocity at perigee changes to $v_2 = \lambda v_1$. (The two orbits are not trivially related by mechanical similarity, since not all lengths are related by the same rescaling.) Since $\ell_2 = \lambda\ell_1$, we have $c_2 = \lambda^2 c_1$ and thus $\epsilon_2 = \lambda^2\epsilon_1 + \lambda^2 - 1 > \epsilon_1$, i.e. the orbit becomes more eccentric for $\lambda > 1$.