1/17/20 Lecture outline

⋆ **Reading: Taylor Chapter 9.**

• Last time: in linearly accelerating frames, $\vec{r}' = \vec{r} - \vec{R}$, get $m\frac{d^2\vec{r}'}{dt^2} = \vec{F} - m\frac{d^2\vec{R}}{dt^2}$.

• Now suppose that the prime frame is rotating relative to the inertial frame, with angular frequency $\vec{\Omega}$. For the earth's rotation $\Omega \approx 7.3 \times 10^{-5} rad/s$. Let $\vec{e}'$ be a unit vector that is fixed in the rotating frame. As seen from the non-rotating frame it has $\frac{d}{dt}\vec{e}' = \Omega \times \vec{e}'$. Recall that a rotation acts on vectors $\vec{k}$ as $k_i' = \sum_{j=1}^{3} R_i^j k_j$ where R is an orthogonal matrix: $R^T = R^{-1}$. For example a rotation around the $\hat{z}$ axis has $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $R^T(\theta) = R(-\theta) = R^{-1}(\theta)$. For $\theta \to d\theta$, then $R = 1 + dR$ with $dR_{21} = -dR_{12} = d\theta$ and all other components vanish. Then $\vec{k}' = \vec{k} + d\vec{k}$ with $d\vec{k} = d\vec{\theta} \times \vec{k}$. Thus $\dot{\vec{k}} = \vec{\Omega} \times \vec{k}$ with $\vec{\Omega} = \dot{\vec{\theta}}$. Also, rotations compose via $R(\theta_1)R(\theta_2)$, which is another rotation; can use this to show that angular velocities add: $\vec{\Omega}_{1+2} = \vec{\Omega}_1 + \vec{\Omega}_2$.

A vector is a geometric object that can be expressed in a basis in either frame $\vec{Q} = \sum_i Q_i \vec{e}_i = \sum_i Q_i' \vec{e}_i'$. The time derivative in the prime frame comes only from that of Q_i' whereas that in the inertial frame includes the time dependence of $\vec{e}_i'$, so $\frac{d\vec{Q}}{dt}|_{inertial} = \frac{d\vec{Q}}{dt}|_{prime} + \vec{\Omega} \times \vec{Q}$.

In particular, $\frac{d\vec{r}}{dt}|_{lab} = \frac{d\vec{r}}{dt}|_{prime} + \vec{\Omega} \times \vec{r}$ and $\frac{d\vec{v}}{dt}|_{lab} = \frac{d\vec{v}}{dt}|_{prime} + \vec{\Omega} \times \vec{v}$, so (if $\vec{\Omega}(t)$ then we get an additional term $\dot{\vec{\Omega}} \times \vec{r}'$)

$$\frac{d^2\vec{r}}{dt^2}|_{lab} = \frac{d^2\vec{r}}{dt^2}|_{prime} + 2\vec{\Omega} \times \frac{d\vec{r}}{dt}|_{prime} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}|_{prime}) = \vec{F}/m.$$

In the prime (or "body") frame we thus have

$$m\frac{d^2\vec{r}'}{dt^2} = \vec{F} + \vec{F}_{cor} + \vec{F}_{cf}, \qquad \vec{F}_{cor} \equiv 2m\dot{\vec{r}}' \times \vec{\Omega}, \quad \vec{F}_{cf} = m\vec{\Omega} \times (\vec{r}' \times \vec{\Omega}).$$

The centrifugal force term can be evaluated using $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$ to give $\vec{F}_{cf} = m\vec{r}'\Omega^2 - m\vec{\Omega}(\vec{r}' \cdot \vec{\Omega})$. For example, consider an observer on the surface of a rotating sphere of radius R (e.g. the earth), with $\vec{\Omega} = \Omega\hat{z}$, then $\vec{F}_{cf} = m\Omega^2 R(\sin\theta\cos\phi, \sin\theta\sin\phi, 0)$.

The Coriolis force term $\vec{F}_{cor} = 2m\dot{\vec{r}}' \times \vec{\Omega}$ is similar to the force of a magnetic field. Taking $\vec{\Omega} = \hat{z}\Omega$, $\vec{F}_{cor}$ is in the (x, y) plane, perpendicular to the velocity in that plane, with direction pointing to the right for the Northern hemisphere (defined to be toward the head of $\vec{\Omega}$), and to the left in the Southern hemisphere.

• Note that the force to the right makes sense for the example of an object sliding on a frictionless, rotating turntable.

• We can see these also in spherical coordinates. Recall $\hat{r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}$, and $\hat{\theta} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z}$, $\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y}$. Then $d\hat{r} = \hat{\theta}d\theta + \sin\theta\hat{\phi}d\phi$, $d\hat{\theta} = -\hat{r}d\theta + \cos\theta\hat{\phi}d\phi$, $d\hat{\phi} = -\sin\theta\hat{r}d\phi - \cos\theta\hat{\theta}d\phi$. Writing out $\frac{d^2\vec{r}}{dt^2}$ in spherical coordinates, $\vec{r} = r\hat{r}$, the terms involving $\dot{\theta}^2$ and $\dot{\phi}^2$ can be interpreted in terms of centrifugal force and those involving $\dot{r}$ and $\dot{\phi}$ or $\dot{\theta}$ can be interpreted in terms of Coriolis force. As a special case (to shorten the formulae), consider $\theta = \pi/2$, where it is the same as for polar coordinates in the plane. Then $\vec{F} = m\frac{d^2\vec{r}}{dt^2} \to F_r = m(\frac{d^2r}{dt^2} - r\dot{\phi}^2)$ and $F_\phi = m(r\frac{d^2\phi}{dt^2} + 2\dot{r}\dot{\phi})$. If we re-write these as equations for $m\frac{d^2r}{dt^2}$ and $mr\frac{d^2\phi}{dt^2}$, the additional terms on the other side of the = sign are interpreted as $F_{r,cf} = mr^2\Omega^2$ and $F_{\phi,cor} = -2m\dot{r}\Omega$.

• Example: free fall near the earth's surface (we omit writing the prime on $\vec{r}'$)

$$m\frac{d^2\vec{r}}{dt^2} = m\vec{g} + 2m\dot{\vec{r}} \times \vec{\Omega}, \qquad \vec{g} = \vec{g}_0 + (\vec{\Omega} \times \vec{r}) \times \vec{\Omega}$$

where $\vec{g}$ is the observed free-fall acceleration, which includes the centrifugal force term. Take $\vec{r} \approx \vec{R}$ in $\vec{F}_{cf}$, which is a vector from the center of the earth to the position on the earth's surface where the experiment is done. Choose local coordinates near that location (assumed to be in the Northern hemisphere) such that $\hat{z}$ points up (it's really $\hat{r}$), $\hat{y}$ points North (it's really $-\hat{\theta}$), and $\hat{x}$ points East (it's really $\hat{\phi}$). In this coordinate system $\vec{\Omega} = (0, \Omega\sin\theta, \Omega\cos\theta)$. This gives

$$\frac{d^2x}{dt^2} = 2\Omega(\dot{y}\cos\theta - \dot{z}\sin\theta), \; \frac{d^2y}{dt^2} = -2\Omega\dot{x}\cos\theta, \; \frac{d^2z}{dt^2} = -g + 2\Omega\dot{x}\sin\theta.$$

Solve this order-by-order in $\Omega \ll 1$. The zero-th order solution is $x^{(0)} = 0$, $y^{(0)} = 0$, $z^{(0)} = h - \frac{1}{2}gt^2$. Plug these into the RHS of the above equation and then solve for the next order; leads to $x^{(1)} = \frac{1}{3}\Omega g t^3 \sin\theta$. So the object falls to the East.