6/1/11 Lecture 19 outline

• As we discussed last time, Einstein’s equations

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}, \]  

are 2nd order, non-linear PDEs for the metric \( g_{\mu\nu} \). Today we’ll briefly discuss solutions to these equations in a linearized approximation, applicable e.g. for discussing gravity waves. We’ll also discuss solutions for \( T_{\mu\nu}^{\text{fluid}} \), applicable for cosmology.

• Expand around flat space, \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \), taking \( h_{\mu\nu} \) small and linearizing in it. So e.g. \( g_{\mu\nu} \approx \eta_{\mu\nu} - h_{\mu\nu} \). We have

\[ \Gamma^\rho_{\mu\nu} \approx \frac{1}{2} \eta^{\rho\sigma} (\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}). \]

And we can drop the \( \Gamma \Gamma \) terms in the Riemann tensor, so

\[ R_{\mu\nu\rho\sigma} \approx \frac{1}{2} (\partial_\rho \partial_\nu h_{\mu\sigma} + \partial_\sigma \partial_\mu h_{\nu\rho} - [\mu \leftrightarrow \nu]). \]

\[ R_{\mu\nu} \approx \frac{1}{2} (\partial_\sigma \partial_\nu h_{\mu\sigma} + \partial_\sigma \partial_\mu h^\sigma_{\mu} - \partial_\mu \partial_\nu h - \partial^2 h_{\mu\nu}), \]

\[ R \approx \partial_\mu \partial_\nu h^{\mu\nu} - \partial^2 h. \]

Plug in to get \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R \).

Can pick names for components, \( h_{00} = -2\Phi \), \( h_{0i} = w_i \), and \( h_{ij} = 2s_{ij} - 2\Psi \delta_{ij} \). Then \( \Gamma^0_{00} = \partial_0 \Phi \), etc. The geodesic equation (taking \( \lambda = \tau/m \) for massive particles)

\[ \frac{dp^\mu}{d\lambda} + \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma = 0 \]

then gives, using \( p^0 = dt/d\lambda = E \) and \( p^i = Ev^i \),

\[ \frac{dp^\mu}{dt} = -\Gamma^\mu_{\rho\sigma} \frac{p^\rho p^\sigma}{E}, \]

or in components

\[ \frac{dE}{dt} = -E (\partial_0 \Phi + 2(\partial_k \Phi) v^k - (\partial_i w_k) - \frac{1}{2} \partial_0 h_{jk} v^j v^k), \]

(giving energy exchange between the particle and gravity) and

\[ \frac{dp^i}{dt} = E [G^i + (\tilde{v} \times H)^i - 2(\partial_0 h_{ij}) v^j - (\partial_{(j} h_{k)i} - \frac{1}{2} \partial_i h_{jk}) v^j v^k] \]
where \( G^i \equiv -\partial_i \Phi - \partial_0 w_i \), and \( H^i \equiv \varepsilon^{ijk} \partial_j w_k \).

- Coordinate transformation, \( \delta h_{\mu\nu} = \partial_{(\mu} \varepsilon_{\nu)} \) similar to gauge transformations in E&M. Can pick convenient gauges, e.g. set \( \Phi = w^i = 0 \). The scalars \( \Phi \) and \( \Psi \) are would-be scalars, but aren’t physical. Neither is the would-be spin 1 component \( w_i \). The only physical dof are the spin \( s = 2 \) quadrupole components \( s_{ij} \). This looks like \( 2s + 1 = 5 \) components, but there’s still more gauge redundancy. Actually, only 2 independent physical polarizations. Counting: \( h_{\mu\nu} \) has 10 polarizations, minus 4 for \( \delta x^\mu = \varepsilon^{\mu}(x) \) symmetry, minus another 4 for the longitudinal condition, gives 2. Gauge symmetry “cuts twice,” like in E&M where we have \( 4 - 1 - 1 = 2 \), here we have \( 10 - 4 - 4 = 2 \).

- Gravity waves in empty space. Take \( T_{\mu\nu} = 0 \) in Einstein’s equations, and linearize them to get \( \partial^2 s_{ij} = 0 \). Call \( h_{TT}^{TT} = 2s_{ij} \) for the \( i, j \) components and zero otherwise. Write a plane wave solution, \( h_{TT}^{TT} = C_{\mu\nu} e^{ikx} \), which solves the wave equation for \( k^2 = 0 \): the graviton is massless. To keep it transverse (eliminate gauge dof), need \( k^\mu C_{\mu\nu} = 0 \). Taking \( k^\mu = (\omega, 0, 0, \omega) \), find, 2 independent polarization components, \( C_{11} = h_+ \) and \( C_{12} = h_X \). A ring of particles in the \( x - y \) plane will oscillate in a \( + \) shape in reaction to a gravitational wave with \( h_+ \neq 0 \), and \( h_X = 0 \). A gravitational wave with \( h_X \neq 0 \) and \( h_+ = 0 \) will cause them to oscillate in a \( X \) pattern. Can define \( h_{R,L} = (h_+ \pm ih_X)/\sqrt{2} \) circular polarizations.

- Gravitational wave production. Let \( I_{ij} = \int d^3x (x,t) x_i x_j \) be the 2nd mass moment. The leading contribution far away, for weak sources, is

\[
h_{ij} - \frac{1}{2} h \delta_{ij} \approx \frac{2}{r} \bar{I}^{ij}(x,t)_{\text{ret}}.
\]

Analogous to \( \vec{A} \sim \vec{p}_{\text{ret}}/r \) in E&M.

LIGO and LISA are laser interferometers, hoping to detect gravity waves.

- Now a bit of cosmology. Consider Einstein’s equations with \( T_{\mu\nu} = T_{\mu\nu}^{\text{fluid}} = (p + \rho) u_\mu u_\nu + pg_{\mu\nu} \). Take \( p = w\rho \), where \( w \) is a constant; where the main cases are \( w = 0 \) for nonrelativistic matter, \( w = \frac{1}{3} \) for massless matter a.k.a. radiation (recall your HW), and \( w = -1 \) for cosmological constant.

Consider first a cosmological constant, with \( \rho = -p \equiv 3\kappa/8\pi G \). Then Einstein’s equations give \( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -3\kappa g_{\mu\nu} \) and a solution of this is \( R_{\rho\sigma\mu}\nu = \kappa(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}) \), which has \( R_{\mu\nu} = 3\kappa g_{\mu\nu} \) and \( R = 12\kappa \). If \( \kappa > 0 \), i.e. positive CC, then the space has positive curvature, and is called deSitter (dS). If \( \kappa < 0 \), i.e. negative CC, then it’s anti-de-Sitter (AdS). These are maximally symmetric spaces.
More interesting spacetimes have the RW form
\[ ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right], \]
where \( a(t) \) is the scale factor and now \( \kappa \) is the curvature of the spatial part of the metric. If \( \kappa = 0 \), the spatial part is flat, while if \( \kappa > 0 \) it has positive curvature (like an \( S^3 \)) so “closed” and if \( \kappa < 0 \) it has negative curvature (hyperbolic space) so “open.”

Can compute the Christoffel connection and curvature of this metric, e.g. \( \Gamma^0_{11} = \frac{a \dot{a}}{1 - \kappa r^2}, \Gamma^3_{03} = \frac{\dot{a}}{a}, \Gamma^1_{11} = \frac{\kappa r}{1 - \kappa r^2} \), etc and \( R_{00} = -\frac{3 \ddot{a}}{a} \) etc., and
\[ R = 6(\ddot{a}/a + (\dot{a}/a)^2 + \kappa/a^2). \]

- Now consider \( T_{\mu\nu} = T_{\mu\nu}^{fluid} = (p + \rho)u_\mu u_\nu + pg_{\mu\nu} \) with this spacetime metric. Note that \( 0 = \nabla_\mu T_{\mu\nu} \), for \( \nu = 0 \) gives \( 0 = -\partial_0 \rho - 3 \frac{\dot{a}}{a} (p + \rho) \). Setting \( p = w\rho \), conservation of energy becomes
\[ \frac{\dot{\rho}}{\rho} = -3(1 + w) \frac{\dot{a}}{a}, \]
so \( \rho_M \sim a^{-3(1+w)} \). For matter, \( w = 0 \), and \( \rho \sim a^{-3} \), i.e. the matter energy density dilutes as the space grows, proportional to the scale factor \( a \), fitting with fixed amount of stuff. For radiation, \( w = \frac{1}{3} \), get \( \rho_R \sim a^{-4} \). This also makes sense: as the space grows \( \sim a \), there is an extra energy density suppression factor of \( a \) compared with the matter case, because the wavelength of the radiation is being redshifted \( \sim a \). Finally, for CC, get \( \rho \sim a^0 \), the energy density of vacuum is independent of the scale factor.

Einstein’s equation for the 00 component gives
\[ \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \frac{\rho}{3 + p}, \]
and for the spatial part \( ij \) gives another equation that leads to the other Friedman equation
\[ \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{\kappa}{a^2}. \]
The Hubble parameter is defined by \( H = \dot{a}/a \) and currently \( H_0 \approx 70 \text{ km/sec/Mpc} \), where \( Mpc = 3 \times 10^{24} \text{ cm} \). In particle physics units, \( H_0 \approx 10^{-33} \text{ eV} \).

Define \( \rho_{crit} = 3H^2/8\pi G \) and \( \Omega = 8\pi G \rho/3H^3 = \rho/\rho_{crit} \), and then \( \Omega - 1 = \kappa/H^2a \).

Write \( \rho_i = \rho_{i0} a^{-n_i} \), where \( n_i = \frac{1}{3} n_i - 1 \), and then the Friedman equation gives \( H^2 = \frac{8\pi G}{3} \sum \rho_i \), where curvature is included as \( \rho_c = -3\kappa/8\pi Ga^2 \), with \( w_i = -\frac{1}{3} \). The Friedman equation gives \( a \sim t^{2/n} \). Matter dominated: \( a = (t/t_0)^{2/3} \); radiation dominated, \( a = (t/t_0)^{1/2} \); vacuum dominated: \( a = e^{H(t-t_0)} \), where \( H = 8\pi \rho/3 = \Lambda/3 \).
If $\Omega_M + \Omega_\Lambda = 1$, then $\kappa = 0$, and the universe is flat. If $\Omega_M + \Omega_\Lambda > 0$ there is positive spatial curvature, $\kappa > 0$. Get

$$\frac{\ddot{a}}{a} = - \frac{4\pi G}{3} (\rho_M - 2\rho_\Lambda).$$

Einstein tried to get a static universe, so he imagined (his self-described “greatest blunder”) that $\rho_M = 2\rho_\Lambda$. If $\rho_M - 2\rho_\Lambda > 0$, the universe decelerates and eventually re-collapses. Observation fits with $2\rho_\Lambda - \rho_M > 0$, so the universe has accelerated expansion. Over time $\rho_M \to 0$, and $\rho_\Lambda = \text{constant}$, so it accelerates more in the future. Observation fits with, presently, $\Omega = 1$: $\Omega_{DE} \approx 75\%, \Omega_{DM} \approx 21\%, \Omega_{NM} \approx 4\%$.

- Consider a scalar field in the RW metric. Its equations of motion become, dropping the spatial derivative terms,

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$$

the second term is “Hubble friction” from the Christoffel connection (can use $\nabla_\mu \nabla^\mu \phi = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \partial^\mu \phi)$ with $\sqrt{-g} = a^3$). The Friedman equation now gives

$$H^2 = \frac{1}{3m_P^2} \left( \frac{1}{2} \dot{\phi}^2 + V(\phi) \right).$$

Inflation is built from such scalars with sufficiently gradual potentials so that the scalar field rolls down very slowly, and the potential looks approximately like a CC. The slow rolling then leads to $H \approx \text{constant}$, an exponential expansion of space.

- Take 225b next quarter for more!