Lorentz tensor redux*

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Contents

1 Introduction

A Lorentz tensor is, by definition, an object whose indices transform like a tensor under Lorentz transformations; what we mean by this precisely will be explained below. A 4-vector is a tensor with one index (a first rank tensor), but in general we can construct objects with as many Lorentz indices as we like.

Recall that we write a 4-vector in components as

$$A^\mu = \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} \tag{1}$$

where we use Greek indices to run over all the spacetime indices, $\mu \in [0, 3]$. We've already seen many examples of 4-vectors in this course; for instance[1],

*As discussed in 4D recitation section on 3/2/15 @UCSD.

[1] We think of 4-vectors with upper indices as column vectors, and 4-vectors with lower indices as row vectors, but when listing the components here we don't worry about organizing them into rows vs. columns.

$$
\begin{aligned}
x^\mu &= (ct, x, y, z) \\
\partial_\mu &= \left(\frac{1}{c}\partial_t, \partial_x, \partial_y, \partial_z\right) \\
k^\mu &= \left(\frac{\omega}{c}, k_x, k_y, k_x\right) \text{ the wave-vector;} \\
j^\mu &= (c\rho, \vec{J}) \text{ for } \rho \text{ the electric charge density,} \\
&\qquad\qquad \vec{J} \text{ the electric current density;} \\
u^\mu &= \frac{dx^\mu}{d\tau} = \gamma\frac{dx^\mu}{dt} = \gamma(c, \vec{v}) \\
p^\mu &= mu^\mu = \left(\frac{E}{c}, \vec{p}\right), \quad E = \gamma mc^2; \\
f^\mu &= \frac{dp^\mu}{d\tau} \\
a^\mu &= \frac{d^2x^\mu}{d\tau^2}
\end{aligned}
$$

We've also seen many conservation equations and examples of Lorentz invariant quantities, like

$$
\begin{aligned}
\partial_\mu j^\mu &= 0 : \text{ current conservation;} \\
\partial^2 A &= \left(\frac{1}{c^2}\partial_t^2 - \vec{\nabla}^2\right) A : \text{ the wave equation, whose form} \\
&\qquad\qquad \text{is invariant under Lorentz transformations;} \\
\Delta s^2 &= (c\Delta t)^2 - (\Delta\vec{x})^2 \text{ the invariant length squared,} \\
&\Rightarrow c\Delta\tau \equiv \sqrt{\Delta s^2} \text{ is a Lorentz scalar, the proper time;} \\
u^2 &= c^2 \\
p^2 &= (mc)^2 = \left(\frac{E}{c}\right)^2 - \vec{p}^2 \\
\sum_{\text{particles}} p^\mu &= (\text{constant})^\mu : \text{ relativistic energy/momentum conservation} \\
&\qquad\qquad \text{for a closed system.}
\end{aligned}
$$

2 The Lorentz transformation

First, we write the components of the Lorentz transformation matrix in index notation. Recall that to transform the components of a 4-vector (let's for now just consider the 4-vector Δx^μ) from an unprimed frame to a frame which is moving at speed v in the $+\hat{x}$ direction relative to F (call it the primed frame), we use the Lorentz transformation

$$
\begin{pmatrix} \Delta x'^0 \\ \Delta x'^1 \\ \Delta x'^2 \\ \Delta x'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta x^0 \\ \Delta x^1 \\ \Delta x^2 \\ \Delta x^3 \end{pmatrix} \tag{2}
$$

where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, $\beta = \frac{v}{c}$. Now we write the components of the Lorentz transformation matrix as $\Lambda^\mu{}_\nu$, where μ is a row index and ν is a column index, such that

$$\Lambda = \begin{pmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & \Lambda^0{}_2 & \Lambda^0{}_3 \\ \Lambda^1{}_0 & \Lambda^1{}_1 & \Lambda^1{}_2 & \Lambda^1{}_3 \\ \Lambda^2{}_0 & \Lambda^2{}_1 & \Lambda^2{}_2 & \Lambda^2{}_3 \\ \Lambda^3{}_0 & \Lambda^3{}_1 & \Lambda^3{}_2 & \Lambda^3{}_3 \end{pmatrix} \tag{3}$$

Then, the Lorentz transformation for Δx^μ can be written in the compact notation

$$\begin{aligned} (\Delta x')^\mu &= \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu \Delta x^\nu \\ &\equiv \Lambda^\mu{}_\nu \Delta x^\nu \end{aligned} \tag{4}$$

We use the Einstein summation convention, meaning that whenever you see two of the same index on one side of an equation, you sum over all the values of that index. The index ν which is summed over lives in the unprimed frame, while the free index μ lives in the primed frame. In particular, by $\Lambda^\mu{}_\nu$ we mean the components of the Lorentz transformation matrix which transforms the components of a 4-vector in a frame associated with the bottom index ν to a frame associated with the top index μ.

For example, for the case of the transformation in Eq. 2, we have $\Lambda^0{}_0 = \Lambda^1{}_1 = \gamma$, $\Lambda^0{}_1 = \Lambda^1{}_0 = -\gamma\beta$, $\Lambda^2{}_2 = \Lambda^3{}_3 = 1$, and the rest of the components are zero. Then

$$\begin{aligned} (\Delta x')^\mu &= \Lambda^\mu{}_0 \Delta x^0 + \Lambda^\mu{}_1 \Delta x^1 + \Lambda^\mu{}_2 \Delta x^2 + \Lambda^\mu{}_3 \Delta x^3 \\ &= \Lambda^\mu{}_0 c\Delta t + \Lambda^\mu{}_1 \Delta x + \Lambda^\mu{}_2 \Delta y + \Lambda^\mu{}_3 \Delta z \\ \\ \Rightarrow (\Delta x')^0 &= (c\Delta t') = \gamma(c\Delta t) - \gamma\beta\Delta x \\ (\Delta x')^1 &= (\Delta x') = -\gamma\beta(c\Delta t) + \gamma\Delta x \\ (\Delta x')^2 &= (\Delta y') = \Delta y \\ (\Delta x')^3 &= (\Delta z') = \Delta z \end{aligned} \tag{5}$$

is the usual Lorentz transformation to a frame moving in the $+\hat{x}$ direction.

The inverse Lorentz transformation should satisfy $(\Lambda^{-1})^\beta{}_\mu \, \Lambda^\mu{}_\nu = \delta^\beta{}_\nu$, where $\delta^\beta{}_\nu \equiv \mathrm{diag}(1,1,1,1)$ is the Kronecker delta. Then, multiply by the inverse on both sides of Eq. 4 to find

$$\begin{aligned} (\Lambda^{-1})^\beta{}_\mu (\Delta x')^\mu &= \delta^\beta{}_\nu \Delta x^\nu \\ &= \Delta x^\beta \end{aligned} \tag{6}$$

The inverse $(\Lambda^{-1})^\beta{}_\mu$ is also written as $\Lambda_\mu{}^\beta$. The notation is as follows: the left index denotes a row while the right index denotes a column, while the top index denotes the frame we're transforming to and the bottom index denotes the frame we're transforming from. Then, the operation $\Lambda_\mu{}^\beta \, \Lambda^\mu{}_\nu$ means sum over the index μ which lives in the primed frame, leaving unprimed indices β and ν (so that the RHS of Eq. 6 is unprimed as it should be), where the sum is over a row of $\Lambda_\mu{}^\beta$ and a column of $\Lambda^\mu{}_\nu$, which is precisely the operation of matrix multiplication.

In particular, one can show that in terms of the components of Λ given in Eq. 3, the components of Λ^{-1} will be given by

$$\Lambda^{-1} = \begin{pmatrix} \Lambda^0{}_0 & -\Lambda^1{}_0 & -\Lambda^2{}_0 & -\Lambda^3{}_0 \\ -\Lambda^0{}_1 & \Lambda^1{}_1 & \Lambda^2{}_1 & \Lambda^3{}_1 \\ -\Lambda^0{}_2 & \Lambda^1{}_2 & \Lambda^2{}_2 & \Lambda^3{}_2 \\ -\Lambda^0{}_3 & \Lambda^1{}_3 & \Lambda^2{}_3 & \Lambda^3{}_3 \end{pmatrix}. \tag{7}$$

As a special case, the inverse to the transformation in Eq. 2 gives

$$\begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta x'^0 \\ \Delta x'^1 \\ \Delta x'^2 \\ \Delta x'^3 \end{pmatrix} = \begin{pmatrix} \Delta x^0 \\ \Delta x^1 \\ \Delta x^2 \\ \Delta x^3 \end{pmatrix} \tag{8}$$

as we expect it to.

3 The metric

The metric $\eta_{\mu\nu}$ is a special Lorentz tensor, which for Minkowski spacetime in our convention is given by

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv \mathrm{diag}(1, -1, -1, -1). \tag{9}$$

The other convention is to use $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, which will change around minus signs in various places. Different spacetime geometries have correspondingly different metrics, but since we live in Minkowski spacetime we only use this form for the metric here.

We use the metric to raise and lower Lorentz indices. By definition $A_\mu \equiv \eta_{\mu\nu} A^\nu$ given a 4-vector A^μ with an upstairs index. Think of A^μ as a column vector, and A_μ as a row vector.

The inverse metric $\eta^{\mu\alpha}$ with upstairs indices satisfies $\eta^{\mu\alpha}\eta_{\alpha\nu} = \delta^\mu{}_\nu$; then, one can show that $\eta^{\mu\alpha} = \mathrm{diag}(1, -1, -1, -1)$. In other words, the Minkowski metric is its own inverse. We can then use the inverse metric to raise indices, as in $A^\mu = \eta^{\mu\alpha} A_\alpha$ given a 4-vector with a lower index. Note that a general metric will not be its own inverse, but the Minkowski metric is a special case.

4 General properties

- We've already seen how to use the metric to raise and lower indices. In general, we use the metric to raise and lower any good Lorentz index. For instance, given a tensor $T_{\alpha\beta}$, we can raise an index by $T^{\mu}_{\ \beta} = \eta^{\mu\alpha}T_{\alpha\beta}$; or given a third-rank tensor $F^{\alpha\beta\gamma}$ we could lower two indices by $F^{\alpha}_{\ \mu\nu} = \eta_{\mu\beta}\eta_{\nu\gamma}F^{\alpha\beta\gamma}$.
- The number of free (unsummed) indices must match on the left-hand side and right-hand side of an equation, while the labels used for indices which are summed over (repeated indices, or "dummy indices") are irrelevant. For instance, $\Delta x^{\mu}\Delta x_{\mu}$ means exactly the same thing as $\Delta x^{\alpha}\Delta x_{\alpha}$ since in both cases we're summing over the index values $[0, 3]$. Also, we've already seen that whether an index appears upstairs or downstairs is important, so in particular the free indices should match in both label and up vs. down placement on both sides of an equation.
- You are guaranteed that an object made up of tensors and 4-vectors with no free indices is Lorentz invariant. We've seen many examples of this before; for instance the invariant interval defined by $\Delta s^2 = \Delta s_{\mu}\Delta s^{\mu} = \eta_{\mu\nu}\Delta s^{\mu}\Delta s^{\nu} = (\Delta s^0)^2 - (\Delta s^1)^2 - (\Delta s^2)^2 - (\Delta s^3)^2 = (c\Delta t)^2 - (\Delta\vec{x})^2$ gives the same number when calculated in any Lorentz frame. In general, we call an object with no free indices a Lorentz scalar.

5 The Lorentz group

With all these ingredients, we can write down the condition for an object Λ to be a Lorentz transformation:

$$\eta_{\mu\nu} = \Lambda^{\alpha}_{\ \mu}\Lambda^{\beta}_{\ \nu}\eta_{\alpha\beta} \tag{10}$$

In the language of matrices, this translates to $\Lambda^T\eta\Lambda = \eta$, for Λ^T the matrix transpose of Λ. As an explicit example, consider the Lorentz transformation of Eq. 2. We note that $\Lambda^T = \Lambda$. We can do out the matrix multiplication, and explicitly verify that:

$$\begin{aligned}\Lambda^T\eta\Lambda &= \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \eta. \end{aligned} \tag{11}$$

This condition is both necessary and sufficient for a 4×4 matrix Λ to leave the inner product of any two 4-vectors invariant. One part of this statement is to show that given any two 4-vectors A^{μ} and B^{μ}, $A \cdot B = A' \cdot B'$:

$$\begin{aligned} A \cdot B = A_\alpha B^\alpha &= \eta_{\alpha\beta} A^\alpha B^\beta; \\ A' \cdot B' = A'_\alpha B'^\alpha &= \eta_{\alpha\beta} A'^\alpha B'^\beta \\ &= \eta_{\alpha\beta}\ (\Lambda^\alpha{}_\mu A^\mu)(\Lambda^\beta{}_\nu B^\nu) \\ &= (\eta_{\alpha\beta}\Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu)\ A^\mu B^\nu \\ &= \eta_{\mu\nu} A^\mu B^\nu = \text{Eq. } 12. \end{aligned} \tag{12}$$

Again, by definition a Lorentz tensor is something whose free indices transform under a Lorentz transformation defined in Eq. 10; for instance, given some tensor $R_{\alpha\beta\gamma}$, in a primed frame one can write the components of R as $R'_{\mu\nu\rho} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \Lambda^\gamma{}_\rho R_{\alpha\beta\gamma}$.

In words what Eq. 10 says is that Lorentz transformations are transformations of spacetime that preserve the Minkowski metric. These form a group, the group of all distance-preserving maps—isometries—of Minkowski spacetime that leave the origin fixed[2].

A group is a set of elements with an operation that combines any two elements to form a third, which satisfies certain properties (closure, associativity, identity, and inverse). Here, the elements are the Λ's and the group operation is matrix multiplication. Then, we have that

- The product of any 2 Lorentz transformations is another Lorentz transformation (closure).
- Associativity of Lorentz transformations (which follows from the properties of matrix multiplication).
- The identity is $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu$ as we've already discussed.
- The inverse of $\Lambda^\mu{}_\nu$ is $(\Lambda^{-1})^\mu{}_\nu = \Lambda_\nu{}^\mu$, as already discussed[3].

Eq. 10 ensures that the laws of physics take the same form in all inertial frames of reference.

[2] The bigger group of isometries of Minkowski spacetime that don't leave the origin fixed is called the Poincaré group. For this group, a coordinate transformation would be given by $x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$ for a^μ a vector of arbitrary constants.

[3] Recall that a square matrix has an inverse if and only if its determinant is nonzero. One can use Eq. 10 to show that the determinant of any Lorentz transformation is nonzero, and thus the inverse always exists.